

# A GEOMETRIC CRITERION FOR THE FINITE GENERATION OF THE COX RING OF PROJECTIVE SURFACES

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**ABSTRACT.** The aim is to give a geometric characterization of the finite generation of the Cox ring of anticanonical rational surfaces. This characterization is encoded in the finite generation of the effective monoid. Furthermore, we prove that in the case of a smooth projective rational surface having a negative multiple of its canonical divisor with only two linearly independent global sections (e.g., an elliptic rational surface), the finite generation is equivalent to the fact that there are only a finite number of smooth projective rational curves of self-intersection  $-1$ . The ground field is assumed to be algebraically closed of arbitrary characteristic.

## 1. INTRODUCTION

In [10], Galindo and Monserrat characterize the smooth projective surfaces  $Z$  defined over an algebraically closed field  $k$  with finitely generated Cox rings (see the next paragraph for the definition) by means of the finiteness of the set of integral curves on  $Z$  of negative self-intersection and the existence of a finitely generated  $k$ -algebra containing two  $k$ -algebras associated naturally to  $Z$ , see [10, Theorem 1, page 94]. The aim of this work is to give an equivalent characterization of the finite generation of the Cox ring totally based on the geometry of the surface and to apply the criterion to some classes of smooth projective rational surfaces, e.g. the anticanonical ones (i.e., those rational surfaces holding an effective anticanonical divisor) and the surfaces constructed in [5], [6], [8] and [9]; establishing thus the geometric nature of our characterization. For some purely algebraic features of the Cox ring of a variety, see [7].

Following Hu and Keel [15], the Cox (or the total coordinate) ring of a smooth projective variety  $V$  defined over an algebraically closed field  $k$  is the  $k$ -algebra defined as follows:

$$\mathrm{Cox}(V) = \bigoplus_{(n_1, \dots, n_r) \in \mathbb{Z}^r} H^0(V, \mathcal{O}(L_1^{n_1} \otimes \dots \otimes L_r^{n_r})).$$

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Here  $(L_1, \dots, L_r)$  is a basis of the  $\mathbb{Z}$ -module  $\text{Pic}(V)$  of classes of invertible sheaves on  $V$  modulo isomorphisms under the tensor product, and we have assumed that the linear and numerical equivalences on the group of Cartier divisors on  $V$  are the same, such assumption is satisfied for example for the smooth projective rational surfaces  $V$ .

An interesting (but still) open problem is to classify theoretically and/or effectively and constructively all smooth projective rational surfaces  $S$  for which the  $k$ -algebra  $\text{Cox}(S)$  is finitely generated. Masayoshi Nagata (see [23]) showed that the surface  $Z$  obtained by blowing up of the projective plane  $\mathbb{P}^2$  at nine or more points in general position has an infinite number of  $(-1)$ -curves (see also [19], [16], [17], [20], [24], [22], [21] and [18] for cases when the points need not be in general position), consequently its Cox ring  $\text{Cox}(Z)$  is not finitely generated. Here a  $(-1)$ -curve on  $Z$  means a smooth projective curve on  $Z$  of self-intersection equal to  $-1$ . Note that in this example, the effective monoid  $M(Z)$  of  $Z$  is also not finitely generated, where  $M(Z)$  stands for the set of elements of the Picard group  $\text{Pic}(Z)$  of  $Z$  having at least a nonzero global section.

In this paper we mainly look for those smooth projective rational surfaces  $S$  for which the finite generation of  $\text{Cox}(S)$  is equivalent to the finite generation of  $M(S)$ . Our two main results, Theorems 1 and 6 below which are derived from Theorem 14, give a partial answer. By the way, we have been informed by a referee that in the characteristic zero case, Theorem 14 was obtained in [1] using a different approach.

**Theorem 1.** *Let  $S$  be a smooth projective rational surface defined over an algebraically closed field  $k$  of arbitrary characteristic such that the invertible sheaf associated to the divisor  $-K_S$  has a nonzero global section.*

*The following assertions are equivalent:*

- (1)  $\text{Cox}(S)$  is finitely generated.
- (2)  $M(S)$  is finitely generated.
- (3)  $S$  has only a finite number of  $(-1)$ -curves and only a finite number of  $(-2)$ -curves.

Here  $K_S$  denotes a canonical divisor on  $S$ .

*Proof.* It follows from Lemma 10 and Theorem 14 below. □

As consequences, the following two results hold:

**Corollary 2.** *The Cox ring of a smooth projective rational surface having a canonical divisor of self-intersection larger than or equal to zero is finitely generated if and only if the set of  $(-1)$ -curves is finite.*

*Proof.* Apply Theorem 1 and [20, Proposition 4.3 (a), page 9]. □

**Corollary 3.** *The Cox ring of a smooth projective rational surface having a canonical divisor of self-intersection larger than zero is finitely generated.*

*Proof.* Apply Theorem 1 and [20, Proposition 4.3 (a), page 9].  $\square$

In particular, since a Del Pezzo surface is nothing but a blow up of the projective plane at  $r$  points with  $r \leq 8$ , we recover the well known result, see [3]:

**Corollary 4.** *The Cox ring of a Del Pezzo surface is finitely generated.*

**Corollary 5.** *The Cox ring of a smooth projective rational surface having an integral curve algebraically equivalent to an anti-canonical divisor is finitely generated if and only if the set of  $(-2)$ -curves is finite and spans a linear subspace in the Picard group of codimension one.*

*Proof.* The result follows from [11] and Theorem 1.  $\square$

Here is our second result:

**Theorem 6.** *Let  $Z$  be a smooth projective rational surface defined over an algebraically closed field  $k$  of arbitrary characteristic such that the invertible sheaf associated to the divisor  $-rK_Z$  has only two linearly independent global sections for some positive integer  $r$ .*

*The following assertions are equivalent:*

- (1)  *$\text{Cox}(Z)$  is finitely generated.*
- (2) *The set of smooth projective rational curves of self-intersection  $-1$  on  $Z$  is finite.*

Here  $K_Z$  denotes a canonical divisor on  $Z$ .

*Proof.* Apply Theorem 14 below and the fact that the set of  $(-2)$ -curves on  $Z$  is finite.  $\square$

## 2. PRELIMINARIES

**2.1. General Notions.** Let  $S$  be a smooth projective surface defined over an algebraically closed field of arbitrary characteristic. A canonical divisor on  $S$ , respectively the Picard group of  $S$  will be denoted by  $K_S$  and  $\text{Pic}(S)$  respectively. There is an intersection form on  $\text{Pic}(S)$  induced by the intersection of divisors on  $S$ , it will be denoted by a dot, that is, for  $x$  and  $y$  in  $\text{Pic}(S)$ ,  $x.y$  is the intersection number of  $x$  and  $y$  (see [14] and [2]).

The following result known as the Riemann-Roch theorem for smooth projective surfaces is stated using the Serre duality.

**Lemma 7.** *Let  $D$  be a divisor on a smooth projective surface  $S$  having an algebraically closed field of arbitrary characteristic as a ground field. Then the*

following equality holds:

$$h^0(S, \mathcal{O}_S(D)) - h^1(S, \mathcal{O}_S(D)) + h^0(S, \mathcal{O}_S(K_S - D)) = 1 + p_a(S) + \frac{1}{2}(D^2 - D.K_S).$$

$\mathcal{O}_S(D)$  (respectively,  $p_a(S)$ ) being an invertible sheaf associated canonically to the divisor  $D$  (respectively, the arithmetic genus of  $S$ , that is  $\chi(\mathcal{O}_S) - 1$ , where  $\chi$  is the Euler characteristic function).

Here we recall some standard results, see [12], [14] and [2]. A divisor class modulo linear equivalence  $x$  of a smooth projective surface  $S$  is effective, respectively numerically effective (nef in short) if an element of  $x$  is an effective, respectively numerically effective, divisor on  $S$ . Here a divisor  $D$  on  $S$  is nef if  $D.C \geq 0$  for every integral curve  $C$  on  $S$ . Now, we start with some properties which follow from a successive iterations of blowing up closed points of a smooth projective rational surface.

**Lemma 8.** *Let  $\pi^* : NS(X) \rightarrow NS(Y)$  be the natural group homomorphism on Néron-Severi groups induced by a given birational morphism  $\pi : Y \rightarrow X$  of smooth projective rational surfaces. Then  $\pi^*$  is an injective intersection-form preserving map of free abelian groups of finite rank. Furthermore, it preserves the dimensions of cohomology groups, the effective divisor classes and the numerically effective divisor classes.*

*Proof.* See [13, Lemma II.1, page 1193]. □

**Lemma 9.** *Let  $x$  be an element of the Néron-Severi group  $NS(X)$  of a smooth projective rational surface  $X$ . The effectiveness or the nefness of  $x$  implies the noneffectiveness of  $k_X - x$ , where  $k_X$  denotes the element of  $Pic(X)$  which contains a canonical divisor on  $X$ . Moreover, the nefness of  $x$  implies also that the self-intersection of  $x$  is greater than or equal to zero.*

*Proof.* See [13, Lemma II.2, page 1193]. □

The following result is also needed. We recall that a  $(-1)$ -curve, respectively a  $(-2)$ -curve, is a smooth rational curve of self-intersection  $-1$ , respectively  $-2$ .

**Lemma 10.** *The monoid of effective divisor classes modulo linear equivalence on a smooth projective rational surface  $X$  having an effective anticanonical divisor is finitely generated if and only if  $X$  has only a finite number of  $(-1)$ -curves and only a finite number of  $(-2)$ -curves.*

*Proof.* See [20, Corollary 4.2, page 109]. □

**2.2. Extremal Surfaces.** Let  $Nef(S)$  denotes the set of nef elements in the Picard group  $Pic(S)$  of a smooth projective surface  $S$ , it has obviously an

algebraic structure as a monoid. We define two more submonoids  $Char(S)$  and  $[Char(S) : Nef(S)]$  of  $Pic(S)$  (see [4] and [10]) as follows:

**Definition 11.** With notation as above.

- (1) The *characteristic monoid*  $Char(S)$  of  $S$  is the set of elements  $x$  in  $Pic(S)$  such that there exists an effective divisor on  $S$  whose associated complete linear system is base point free and whose class in  $Pic(S)$  is equal to  $x$ .
- (2) The *monoid of fractional base point free effective classes*  $[Char(S) : Nef(S)]$  of  $S$  is the set of elements  $y$  in  $Pic(S)$  such that there exists a positive integer  $n$  with  $ny \in Char(S)$ .

The main properties that we are interested in regarding  $Char(S)$  and  $[Char(S) : Nef(S)]$  of a smooth projective surface are the ones in the Lemma below. Their proofs are straightforward.

**Lemma 12.** *With notation as above, the followings hold:*

- (1)  $Char(S)$  and  $[Char(S) : Nef(S)]$  are submonoids of  $Nef(S)$ .
- (2)  $Char(S) \subseteq [Char(S) : Nef(S)]$ .

Here we define the ingredient needed for our criterion:

**Definition 13.** With notation as above,  $S$  is extremal if the monoid of fractional base point free effective classes  $[Char(S) : Nef(S)]$  is maximal, that is, if  $Nef(S) = [Char(S) : Nef(S)]$ .

### 3. THE CRITERION

Now we are able to state our geometric criterion:

**Theorem 14.** *Let  $S$  be a smooth projective surface defined over an algebraically closed field of arbitrary characteristic. The following assertions are equivalents:*

- (1) The Cox ring  $Cox(S)$  is finitely generated.
- (2)  $S$  satisfies the following two properties:
  - i.  $S$  is extremal, and
  - ii. the effective monoid  $M(S)$  of  $S$  is finitely generated.
- (3)  $S$  satisfies the following two properties:
  - i.  $S$  is extremal, and
  - ii. the nef monoid  $Nef(S)$  of  $S$  is finitely generated.

*Proof.* By duality, it is obvious that (2) is equivalent to (3). Assume that the Cox ring  $Cox(S)$  of  $S$  is finitely generated, it follows that the effective monoid  $M(S)$  of  $S$  is finitely generated. On the other hand, if  $y$  is an effective element of  $Nef(S)$ , then if  $y$  belongs to  $Char(S)$ , we are done and if not we

may find some positive integer  $s$  such that  $sy$  is an element of  $\text{Char}(S)$ . Hence  $[\text{Char}(S) : \text{Nef}(S)]$ , i.e.,  $S$  is extremal. Conversely, if  $S$  is extremal, and the nef monoid  $\text{Nef}(S)$  of  $S$  is finitely generated, then it follows from [10] that the Cox ring of  $S$  is finitely generated.  $\square$

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